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Polynomials With Large Partial Sums*

D. J. NEWMAN

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

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If $|a_0 + a_1 z + \cdots a_N z^N| \leqslant 1$ throughout $|z| \leqslant 1$, how large can $\text{Max}_k |a_0 + a_1 + \cdots a_k|$ be?

$$|a_0 + a_1 + \cdots a_k| \leqslant \sum_{\nu=0}^{k} (\nu^{-1/2})^2 = \frac{\log k}{\pi} + O(1) \leqslant \frac{\log N}{\pi} + O(1),$$

but since this theorem is not sharp for polynomials we cannot conclude that the quantity in question ever gets this large. Our purpose is to find the *exact* bound for the "middle" partial sum and so thereby to establish $(\log N)/\pi + O(1)$ as the correct value. The precise statement is the following:

THEOREM. *Over all polynomials, $a_0 + a_1 z + a_2 z^2 + \cdots + a_{2n-1} z^{2n-1}$, which are bounded by* 1 *throughout the unit disk, the* maximum *value of* $|a_0 + a_1 + \cdots a_{n-1}|$ *is* exactly

$$\frac{1}{2} + \frac{1}{2n} \sum_{k=1}^{n} \csc \frac{\pi(2k-1)}{2n}.$$

LEMMA 1. *If $P(z) = a_0 + a_1 z + \cdots + a_{2n-1} z^{2n-1}$ and $S = a_0 + a_1 + \cdots + a_{n-1} - a_n - a_{n+1} - \cdots - a_{2n-1}$, then*

$$S = \frac{2}{n} \sum_{\omega^n = -1} \frac{P(\omega)\,\omega}{\omega - 1}.$$

Proof. This obviously follows, by addition, from the two special cases

$$\text{(A)} \quad \deg P(z) < n, \qquad \text{and} \qquad \text{(B)} \quad 1 + z^n \mid P(z).$$

Indeed (A) follows from the partial decomposition

$$\frac{P(z)}{1 + z^n} = \frac{1}{n} \sum_{\omega^n = -1} \frac{P(\omega)\,\omega}{\omega - z},$$

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upon setting $z = 1$ and noting that $S = P(1)$. As for (B), on the other hand all the $P(\omega) = 0$ so that the right side vanishes, but since $P(z) = (1 + z^n) \times (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1})$ it follows that $a_n = a_0$, $a_{n+1} = a_1, \ldots$, etc., and so $S = 0$ also.

LEMMA 2. *If* $L_\omega(z) = ((z^n + 1)/n)(\omega/(\omega - z))$ *(the Lagrange interpolators) then* $\sum_{\omega^n=-1} | L_\omega(z)|^2 = 1$ *identically on* $| z | = 1$.

Proof. Note first that

$$L_\omega(z) = (\omega/n)(\omega^{n-1} + \omega^{n-2} z + \cdots + z^{n-1}) \leqslant (1/n)(1 + 1 + \cdots + 1) = 1$$

so that $L_\omega(z)$ has a double 1 point at $z = \omega$. Since all the other terms vanish (doubly) at this point it follows that $\sum_{\omega^n=-1} | L_\omega(z)|^2 - 1$ has a double zero at each of the nth roots of -1. When z is set equal to $e^{i\theta}$, howefer, this expression becomes a trigonometric polynomial of degree $n - 1$. Since it has $2n$ zeros (counting multiplicity) it follows that it is identically 0.

Proof of the theorem. Lemma 1 gives

$$\begin{aligned} a_0 + a_1 + \cdots + a_{n-1} &= \frac{P(1)}{2} + \frac{1}{n} \sum_{\omega^n=-1} P(\omega) \frac{\omega}{\omega - 1} \\ &\leqslant \frac{1}{2} + \frac{1}{n} \sum_{\omega^n=-1} \frac{1}{| \omega - 1 |} \\ &= \frac{1}{2} + \frac{1}{2n} \sum_{k=1}^{n} \csc \frac{\pi(2k-1)}{2n}. \end{aligned}$$

This supplies the upper bound and we show that it is the *exact* bound by producing a $P(z)$ for which the $\leqslant$ becomes $=$. Thus we must construct a $P(z)$ of degree $2n - 1$ for which

$$| P(z)| \leqslant 1 \qquad \text{througout } | z | = 1, \tag{1}$$

$$P(\omega) = (i/\omega^{1/2})(\mathrm{I\,m}\, \omega^{1/2} > 0) \qquad \text{for all } \omega,\ \omega^n = -1, \tag{2}$$

$$P(1) = 1. \tag{3}$$

Indeed, following the notation of Lemma 2, we begin by setting

$$p(z) = \sum_{\omega^n=-1} \frac{(z + \omega)}{2\omega(\omega^{1/2})} L_\omega^2(z) \qquad (\deg p \leqslant 2n - 1), \tag{4}$$

and we observe, by that lemma, that

$$| p(z)| \leqslant 1 \qquad \text{throughout } | z | \leqslant 1. \tag{5}$$

Also observe that

$$p(\omega) = 1/\omega^{1/2} \qquad \text{for all the } \omega. \tag{6}$$

Next we note that the terms comprising $p(1)$ may be paired off, ω with $1/\omega$ (possibly with one additional term, 0, if $\omega = -1$) and these pairs add to

$$\frac{1}{2n^2}\left[\frac{(\omega+1)}{\omega(\omega^{1/2})}\frac{\omega^2}{(\omega-1)^2} + \frac{\omega^{-1}+1}{(1/\omega)(1/\omega)^{1/2}}\frac{\omega^{-2}}{(1/\omega-1)^2}\right],$$

which is 0 since $(1/\omega)^{1/2} = -1/\omega^{1/2}$ by our convention requiring $\operatorname{Im} \omega^{1/2} > 0$. We conclude that

$$p(1) = 0. \tag{7}$$

Because of (5), Shapiro's theorem [2] produces a $q(z)$ with

$$|p(z)|^2 + |q(z)|^2 \equiv 1 \qquad \text{all along } |z| = 1, \qquad \deg q \leqslant 2n-1. \tag{8}$$

Furthermore, we may normalize so that $q(1) \geqslant 0$, in which case, by (7) and (8), we obtain

$$q(1) = 1. \tag{9}$$

By (6), however, all the $|p(\omega)| = 1$ and so (8) gives $q(\omega) = 0$ or $z^n + 1 \mid q(z)$. Thus we may write the partial fraction decomposition

$$\frac{q(z)}{(z^n+1)^2} = \sum_{\omega^n=-1} \frac{\alpha_\omega}{z-\omega},$$

and if, further, we set $\alpha_\omega = \omega^{1/2}(a_\omega + ib_\omega)$, a and b all real, and call

$$r(z) = -(z^n+1)^2 \sum_{\omega^n=-1} \frac{i(\omega^{1/2})\, a_\omega}{z-\omega}, \quad s(z) = (z^n+1)^2 \sum_{\omega^n=-1} \frac{i(\omega^{1/2})\, b_\omega}{z-\omega}, \tag{10}$$

we conclude that

$$q(z) = ir(z) + s(z). \tag{11}$$

The crucial observation is the fact that

all the terms in $p(z)$, $s(z)$, or $r(z)$ are *real multiples* of $z^{n-1/2}$. (12)

Among other things this says that at $z = 1$ both $r(z)$ and $s(z)$ are real and so (9) and (11) give

$$s(1) = 1. \tag{13}$$

But moreover, (12) shows us that on $|z| = 1$,

$$|q(z)|^2 \equiv |r(z)|^2 + |s(z)|^2, \qquad |s(z) + ip(z)|^2 \equiv |s(z)|^2 + |p(z)|^2. \tag{14}$$

We conclude from (14) and (8) that, on $|z| = 1$,

$$|s(z)| + |p(z)|^2 + |r(z)|^2 \equiv 1, \tag{15}$$

and so the choice $P(z) = s(z) + ip(z)$ satisfies (1), (2), and (3). Namely, (1) follows from (15), (2) follows from (6) and (10), and (3) follows from (7) and (13). The proof is complete.

The exact values for other than the "middle" partial sums seem difficult to obtain. In fact, this middle one is not the maximizing partial sum, even for third-degree polynomials. For general even-degree polynomials we cannot obtain any explicit formulas, although for $n = 2$ it is a simple exercise to show that $2 + 4z - z^2$ is the extremal. It has maximum modulus $3(3^{1/2})$ and maximum partial sum 6 (for ratio of $2/3^{1/2}$).

Despite all the open questions, however, we point out that our quantity

$$\frac{1}{2} + \frac{1}{2n}\sum_{k=1}^{n} \csc\frac{\pi(2k-1)}{2n} \text{ is } \frac{\log(2n-1)}{\pi} + O(1) \qquad \left(= \frac{\log 2n}{\pi} + O(1)\right),$$

and this is a lower bound for the maximum partial sum. Combining this with Landau's theorem, therefore, gives the (asymptotic) answer, $(\log N)/\pi + O(1)$, to our original question.

References

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2. H. S. Shapiro, Doctoral Thesis, M.I.T., Cambridge, Mass., June 1952.